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Vorticity–strain residual-based turbulence modelling of the Taylor–Green vortex

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SUMMARY

This paper describes turbulence properties for a residual-based turbulence model using fine-scale approximations of strain and vorticity, emanating from a split of the non-linear term in the Navier–Stokes equations. We will discuss the modelling of the fine-scale terms and their impact in the light of simulation results of turbulent transition of the Taylor–Green vortex. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Efficient methods for the simulation of complex turbulent flows, governed by the Navier–Stokes equations, are still a challenging task in scientific computing and engineering. The difficulty lies in the seemingly random behaviour of the flow and the broad range of scales present, going from the size of the physical objects, which can be of the order of several meters, down to submillimeter sized eddies. For problems where only the mean quantities are of interest, such as drag and lift, methods considering the ensemble average of the flow, the so-called Reynolds averaged Navier–Stokes equations, are often enough. In cases where higher frequency phenomena are studied, such as flow-induced noise, one needs to use large eddy simulation (LES) techniques, where not only the average flow field is resolved but also larger unsteady structures are included.

The general approach to derive the LES equations is to apply a low-pass filter to the Navier–Stokes equations, in which case an unknown subgrid scale stress tensor appears in the equations that needs to be modelled. An alternative way is to consider the numerical truncation as a filtering operation and use the implicit numerical error to represent the effect of the unresolved scales. For an overview on the background of LES and different approaches to model the subgrid scales, see the book by Sagaut [1] and the references therein.

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This paper deals with an approach to subgrid scale modelling derived in the framework of the variational multiscale (VMS) method, see Hughes [2] and Hughes *et al.* [3], together with using residuals when computing the fine-scale behaviour. The VMS method provides a general framework for derivation of multiscale methods where the effects of fine (unresolved) scales on coarse (resolved) scale are accounted for through additional terms in the variational formulation. The additional terms involve the fine-scale part of the solution which satisfies an equation driven by the residual of the coarse-scale part of the solution. In the simplest case, the fine scales can be approximated by a suitable scaling of the residual. This approach is closely related to fine scales approximated by element bubble functions, see Brezzi *et al.* [4] and Arbogast [5]. While derived in a completely different setting, similar ideas are present in the two-level simulation model by Kemenov and Menon [6]. There are also examples of stabilization techniques based on multiscale ideas such as the one by Codina [7] and the selective stabilization technique by Tezduyar and Sathe [8], which allows different stabilization to be applied on different scales. Applications of multiscale techniques to turbulence modelling appear in Hughes *et al.* [9] and Hughes and Oberai [10], where a modified Smagorinsky model only acting on the fine-scale part of the solution was applied, and in Calo [11], Hughes *et al.* [12], and Scovazzi [13], where a residual-based turbulence model was introduced.

The specific formulation used in this paper is based on splitting the non-linear term in the Navier–Stokes equations into vorticity and strain, and apply the multiscale modelling to these two quantities instead of velocity or velocity gradient as is done in, e.g. [11–13]. The rationale is that we believe it may be easier to capture phenomenon such as vorticity stretching, which is important in turbulence dynamics, when this splitting is used. Moreover, Kerr *et al.* [14] show that vorticity is an important mechanism in the turbulent energy transfer process. It is shown, both in theory and in numerical experiments, that the effects of unresolved subgrid scales in this formulation are not necessarily stabilizing but on the contrary may drive certain important phenomena in the flow. This is expected since the discretization often introduce artificial viscosity that shadows or kills important flow structures, and a proper description of subgrid scale effects should thus compensate for such effects. This is generally not the case for contemporary turbulence models. Also, key to these results is that we do not start to model the effects of unresolved scales directly in the partial differential equation setting, as is customary in traditional approaches to turbulence modelling, but instead take the effect of the numerical discretization into account in our approximation of the subgrid scales. The application studied in this paper is the Taylor–Green vortex problem, where an initially low-frequency sinusoidal velocity field undergoes transition to turbulence. This is a challenging case due to the need to correctly capture the turbulent production in this statistically unsteady flow.

The outline of the paper is as follows: In Section 2, we introduce a VMS framework for the Navier–Stokes equations, in Section 3 we discuss residual-based approximations of the fine-scale contributions, in Section 4 we study the effects of these terms on the Taylor–Green vortex, and finally in Section 5 we discuss our results and future research directions.

2. THE VARIATIONAL MULTISCALE FRAMEWORK FOR NAVIER-STOKES EQUATIONS

2.1. The Navier–Stokes equations

In this paper, we shall consider the flow of an incompressible fluid in a cube $\Omega \subset \mathbf{R}^3$ with periodic boundary conditions in all three directions. The flow is governed by the Navier–Stokes

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equations: find the velocity $\mathbf{u} : \Omega \to \mathbf{R}^3$ and pressure $p : \Omega \to \mathbf{R}$ such that

$$\dot{\mathbf{u}} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \Delta \mathbf{u} = \mathbf{0} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

together with periodic boundary conditions in all three directions and given initial velocity field $\mathbf{u} = \mathbf{u}_0$ at time $t = 0$. We shall, in particular, consider the so-called Taylor–Green vortex corresponding to the initial conditions

$$\begin{aligned} u_{1,0} &= \sin(x)\cos(y)\cos(z) \\ u_{2,0} &= -\cos(x)\sin(y)\cos(z) \\ u_{3,0} &= 0 \end{aligned} \tag{3}$$

together with a divergence-free pressure field computed *via* the Poisson equation on the cube $\Omega = [0, 2\pi]^3$. The Reynolds number is defined as $Re = UL/\nu$, where, for this set-up, the velocity scale $U = 1$ and the length scale $L = 1$, and ν is the kinematic viscosity. This flow is a simple yet challenging problem involving vortex stretching and production of small-scale eddies.

We next introduce the notation $\mathscr{V}_{\mathbf{u}} = [H^1(\Omega)]^3$, $\mathscr{V}_p = L^2(\Omega)$ for the velocity and pressure spaces, respectively, and $\mathscr{V} = \mathscr{V}_{\mathbf{u}} \times \mathscr{V}_p$ for the combined space. The variational form of (1) reads: find $(\mathbf{u}, p) : [0, t] \to \mathscr{V}$ such that

$$(\dot{\mathbf{u}}, \mathbf{v}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (\nabla \cdot \mathbf{u}, q) + (\nu \nabla \mathbf{u}, \nabla \mathbf{v}) = 0 \tag{4}$$

for all $(\mathbf{v}, q) \in \mathscr{V}$. Since we consider an incompressible flow, we can use the following identity:

$$\boxed{\mathbf{u} \cdot \nabla \mathbf{u} = S\mathbf{u} + \mathbf{\Omega} \times \mathbf{u}} \tag{5}$$

where $S = (\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}})/2$ is the strain tensor and $\mathbf{\Omega} = \boldsymbol{\omega}/2 = (\nabla \times \mathbf{u})/2$ is (half) the vorticity, and rewrite the variational statement in the form: find $(\mathbf{u}, p) : [0, t] \to \mathscr{V}$ such that

$$(\dot{\mathbf{u}}, \mathbf{v}) + (S\mathbf{u}, \mathbf{v}) + (\mathbf{\Omega} \times \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (\nabla \cdot \mathbf{u}, q) + (\nu \nabla \mathbf{u}, \nabla \mathbf{v}) = 0 \tag{6}$$

for all $(\mathbf{v}, q) \in \mathscr{V}$, which brings out the action of the strain and vorticity.

2.2. *The vorticity–strain VMS*

In order to reach a computationally affordable solution we need to remove the smallest scales in the flow and only compute the solution for larger-scale flow features. The approach chosen here is based on using variational arguments to split the problem into different scales where the small-scale solution is approximated or modelled and its impact on the large-scale equations is computed, see Hughes *et al.* [3] for an overview of the basic ideas. Choose two spaces $\mathscr{V}_{\mathrm{c}} \subset \mathscr{V}$ and $\mathscr{V}_{\mathrm{f}} \subset \mathscr{V}$ such that

$$\mathscr{V} = \mathscr{V}_{\mathrm{c}} \oplus \mathscr{V}_{\mathrm{f}} \tag{7}$$

where $\mathscr{V}_{\mathrm{c}}$ is associated with the coarse (large) scale and $\mathscr{V}_{\mathrm{f}}$ is associated with the fine (small) scale. That is, $\mathscr{V}_{\mathrm{c}}$ represents the space where we seek a numerical solution to our flow, and $\mathscr{V}_{\mathrm{f}}$ spans

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the subgrid scales which are subject to approximation or modelling. Thus $\mathscr{V}_\mathrm{c}$ typically consists of piecewise polynomial finite element functions.

Introducing these spaces in (4) gives us the following weak formulation: find $(\mathbf{u}_\mathrm{c}, p_\mathrm{c}) \in \mathscr{V}_\mathrm{c}$ and $(\mathbf{u}_\mathrm{f}, p_\mathrm{f}) \in \mathscr{V}_\mathrm{f}$ such that

$$\begin{aligned} 0 = (\dot{\mathbf{u}}_\mathrm{c} + \dot{\mathbf{u}}_\mathrm{f}, \mathbf{v}) + ((S_\mathrm{c} + S_\mathrm{f})(\mathbf{u}_\mathrm{c} + \mathbf{u}_\mathrm{f}), \mathbf{v}) + ((\mathbf{\Omega}_\mathrm{c} + \mathbf{\Omega}_\mathrm{f}) \times (\mathbf{u}_\mathrm{c} + \mathbf{u}_\mathrm{f}), \mathbf{v}) \\ - (p_\mathrm{c} + p_\mathrm{f}, \nabla \cdot \mathbf{v}) + (\nabla \cdot (\mathbf{u}_\mathrm{c} + \mathbf{u}_\mathrm{f}), q) + (\nu \nabla (\mathbf{u}_\mathrm{c} + \mathbf{u}_\mathrm{f}), \nabla \mathbf{v}) \end{aligned} \tag{8}$$

for all $(\mathbf{v}, q) = (\mathbf{v}_\mathrm{c}, q_\mathrm{c}) \in \mathscr{V}_\mathrm{c}$ and $(\mathbf{v}, q) = (\mathbf{v}_\mathrm{f}, q_\mathrm{f}) \in \mathscr{V}_\mathrm{f}$. This separates the scales in the equations into two parts. Choosing coarse-scale test functions we get the coarse-scale equation where the fine-scale contributions account for the effects of the fine scales on the coarse scales, and *vice versa*.

The energy exchange between scales in Navier–Stokes equations mainly occurs in the non-linear term and thus we focus on the fine-scale information contained in the non-linear terms involving the fine-scale strain S_f, the fine-scale vorticity $\mathbf{\Omega}_\mathrm{f}$, and the fine-scale velocity $\mathbf{u}_\mathrm{f}$. We note that although there is a relation between S and $\mathbf{\Omega}$ in an incompressible flow, the discrete solution is not divergence free and thus both terms need to be included. Rewriting the non-linear term once more to bring out the action of these three fine-scale quantities we get

$$\begin{aligned} &((S_\mathrm{c} + S_\mathrm{f})(\mathbf{u}_\mathrm{c} + \mathbf{u}_\mathrm{f}), \mathbf{v}) + ((\mathbf{\Omega}_\mathrm{c} + \mathbf{\Omega}_\mathrm{f}) \times (\mathbf{u}_\mathrm{c} + \mathbf{u}_\mathrm{f})) \\ &\quad = (S_\mathrm{c}\mathbf{u}_\mathrm{c} + \mathbf{\Omega}_\mathrm{c} \times \mathbf{u}_\mathrm{c}) + (S_\mathrm{f}\mathbf{u}_\mathrm{c} + \mathbf{\Omega}_\mathrm{f} \times \mathbf{u}_\mathrm{c}) + (S_\mathrm{c}\mathbf{u}_\mathrm{f} + \mathbf{\Omega}_\mathrm{c} \times \mathbf{u}_\mathrm{f}) + (S_\mathrm{f}\mathbf{u}_\mathrm{f} + \mathbf{\Omega}_\mathrm{f} \times \mathbf{u}_\mathrm{f}) \\ &\quad = \mathbf{u}_\mathrm{c} \cdot \nabla \mathbf{u}_\mathrm{c} + (S_\mathrm{f}\mathbf{u}_\mathrm{c} + \mathbf{\Omega}_\mathrm{f} \times \mathbf{u}_\mathrm{c}) + \mathbf{u}_\mathrm{f} \cdot \nabla \mathbf{u}_\mathrm{c} + (S_\mathrm{f}\mathbf{u}_\mathrm{f} + \mathbf{\Omega}_\mathrm{f} \times \mathbf{u}_\mathrm{f}) \end{aligned} \tag{9}$$

The first of these terms is the ordinary coarse-scale non-linear term, the second the interaction between coarse-scale transport and fine-scale vorticity and strain, the third the fine-scale transport of the coarse-scale velocity gradient, and the fourth fine-scale/fine-scale interaction. There are several modelling alternatives possible for the three terms involving the unresolved fields. Here we choose to apply quasi-stationary approximations of the evolution of the fine-scale vorticity, strain, and velocity, respectively, and study their impact on transition to turbulence. Some alternative modelling approaches will be briefly discussed in the concluding remarks of the paper.

Without going into details in theoretical analysis of this method, some insight on possible benefits of this formulation can be gained from the following simple arguments. Expanding the vorticity in terms of the orthonormal eigenvectors ξ_i, with corresponding eigenvalues λ_i, of the strain tensor S_c gives

$$\mathbf{\Omega} + \tau S_\mathrm{c} \mathbf{\Omega} = \sum_{i=1}^{3} (1 + \tau_{\mathbf{\Omega}} \lambda_i) \omega_i \xi_i \tag{10}$$

Recalling that $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we see that there is always at least one positive and one negative eigenvalue. Thus, the modified vorticity aligns with the eigenvector associated with the largest eigenvalue of the strain tensor leading to a stronger tendency for vortex stretching, a physical phenomenon believed to be important in the creation of turbulence. This motivates the somewhat intuitive comment on possible advantages of using a vorticity–strain representation of the velocity gradient in the introduction. Moreover, Kerr *et al.* [14] performs an *a priori* analysis of the subgrid scale energy transfer in isotropic turbulence, which indicates the importance of vorticity–velocity interactions, both with respect to correlation with the traditional subgrid stress tensor and with respect to the transfer of kinetic energy between coarse and fine scales.

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Thus, assuming access to computable approximations of $\mathbf{\Omega}_f = \mathbf{\Omega}_f(\mathbf{u}_c, p_c)$, $S_f = S_f(\mathbf{u}_c, p_c)$, and $\mathbf{u}_f = \mathbf{u}_f(\mathbf{u}_c, p_c)$ given $(\mathbf{u}_c, p_c)$ we get the following vorticity–strain VMS method: find $(\mathbf{u}_c, p_c) : [0, t] \rightarrow \mathscr{V}_c$ such that

$$\begin{aligned} 0 = {} & (\dot{\mathbf{u}}_c, \mathbf{v}_c) + ((\mathbf{u}_c + \mathbf{u}_f) \cdot \nabla \mathbf{u}_c, \mathbf{v}_c) + (S_f \mathbf{u}_c + \mathbf{\Omega}_f \times \mathbf{u}_c, \mathbf{v}_c) \\ & + (S_f \mathbf{u}_f + \mathbf{\Omega}_f \times \mathbf{u}_f, \mathbf{v}_c) - (p_c, \nabla \cdot \mathbf{v}_c) + (\nabla \cdot \mathbf{u}_c, q_c) + (\nu \nabla \mathbf{u}_c, \nabla \mathbf{v}_c) \end{aligned} \tag{11}$$

for all $(\mathbf{v}_c, q_c) \in \mathscr{V}_c$. To close this equation, we need to specify how $S_f(\mathbf{u}_c, p_c)$, $\mathbf{\Omega}_f(\mathbf{u}_c, p_c)$, and $\mathbf{u}_f(\mathbf{u}_c, p_c)$ are to be computed.

3. RESIDUAL-BASED MODELLING OF THE FINE-SCALE STRAIN, VORTICITY, AND VELOCITY

Since the aim is to reduce the computational complexity compared with direct numerical simulation (DNS) of the Navier–Stokes equations, affordable approximations of S_f, $\mathbf{\Omega}_f$, and u_f must be constructed. Starting with the evolution equations for strain, vorticity, and velocity, respectively, we will simplify these to algebraic, ordinary differential equations. Furthermore, quasi-static conditions for the fine scales are assumed, which further reduces the complexity and lead to an almost linear scaling of the coarse-scale residuals. This yields a simple and cheap fine-scale approximation, although very crude. Despite this, the computational experiments indicate that some interesting features are captured, but they also show a need for a more careful approach in order to bring out more aspects of the physics involved. One reason to the good performance of these approximations might be that it has been shown, see [14, 15], that the largest unresolved scales are the most important and it is reasonable to believe that they are related to the residuals.

3.1. Evolution of fine-scale vorticity

We begin with the approximation of the fine-scale vorticity $\mathbf{\Omega}_f$. Taking the curl of the momentum equation (1) we obtain the following equation for the vorticity:

$$\dot{\mathbf{\Omega}} + \mathbf{u} \cdot \nabla \mathbf{\Omega} = S\mathbf{\Omega} + \nu \Delta \mathbf{\Omega} \tag{12}$$

Setting $\mathbf{\Omega} = \mathbf{\Omega}_c + \mathbf{\Omega}_f$, $S \approx S_c$, and $\mathbf{u} \approx \mathbf{u}_c$ we get an approximate equation for the evolution of the fine-scale vorticity $\mathbf{\Omega}_f$

$$(\dot{\mathbf{\Omega}}_f + \mathbf{u}_c \cdot \nabla \mathbf{\Omega}_f - S_c \mathbf{\Omega}_f + \nu \Delta \mathbf{\Omega}_f, \mathbf{v}_f) = (R_\omega(\mathbf{\Omega}_c, S_c, \mathbf{u}_c), \mathbf{v}_f) \tag{13}$$

where $R_\omega(\mathbf{\Omega}_c, S_c, \mathbf{u}_c) = -\dot{\mathbf{\Omega}}_c - \mathbf{u}_c \cdot \nabla \mathbf{\Omega}_c + S_c \mathbf{\Omega}_c + \nu \Delta \mathbf{\Omega}_c$ is the coarse-scale vorticity residual.

Following Codina [7] and Codina *et al.* [16], the left-hand side is approximated as follows:

$$\left(\dot{\mathbf{\Omega}}_f + \frac{1}{\tau_\omega} \mathbf{\Omega}_f, v_f \right) = (R_\omega(\mathbf{\Omega}_c, S_c, \mathbf{u}_c), v_f) \tag{14}$$

where $1/\tau_\omega = (c_1 |\mathbf{u}_c| / h + c_2 \nu / h^2)$ with constants c_1 and c_2. According to the mean value theorem for Fourier series, there exists constants c_1 and c_2 such that the solution to (14) has the same element-wise L^2-norm as the solution to (13). The coarse-scale strain acting on $\mathbf{\Omega}_f$ has been neglected in τ_ω since it appears to be small compared with $|\mathbf{u}_c|/h$ and ν/h^2.

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Assuming a quasi-stationary behaviour of the fine-scale fields, i.e. $\dot{\mathbf{\Omega}} \approx 0$, we finally obtain the following expression for $\mathbf{\Omega}_\mathrm{f}$:

$$\boxed{\mathbf{\Omega}_\mathrm{f} \approx \tau_\omega R_\omega(\mathbf{\Omega}_\mathrm{c})} \tag{15}$$

3.2. Evolution of fine-scale strain

Repeating the arguments, we have that the strain tensor S satisfies the equation, see [17]

$$\dot{S} + \mathbf{u}\cdot\nabla S = -S^\mathrm{T} S - (\mathbf{\Omega}\mathbf{\Omega}^\mathrm{T} - I\mathbf{\Omega}\cdot\mathbf{\Omega})/4 - \mathscr{H}(p) + \nu\Delta S \tag{16}$$

where I is the identity matrix and $\mathscr{H}(p)$ is the pressure Hessian. Setting $S = S_\mathrm{c} + S_\mathrm{f}$, $\mathbf{u} \approx \mathbf{u}_\mathrm{c}$, and $\mathbf{\Omega} = \mathbf{\Omega}_\mathrm{c} + \mathbf{\Omega}_\mathrm{f} \approx \mathbf{\Omega}_\mathrm{c} + \tau_\omega R_\omega(\mathbf{\Omega}_\mathrm{c})$, we get the following equation for the evolution of fine-scale strain tensor S_f:

$$\begin{aligned}
&\dot{S}_\mathrm{f} + \mathbf{u}_\mathrm{c}\cdot\nabla S_\mathrm{f} + 2S_\mathrm{c}^\mathrm{T} S_\mathrm{f} + S_\mathrm{f}^\mathrm{T} S_\mathrm{f} + \mathscr{H}(p_\mathrm{f}) - \nu\Delta S_\mathrm{f} \\
&\quad = -\dot{S}_\mathrm{c} - \mathbf{u}_\mathrm{c}\cdot\nabla S_\mathrm{c} - S_\mathrm{c}^\mathrm{T} S_\mathrm{c} + \nu\Delta S_\mathrm{c} - (\mathbf{\Omega}\mathbf{\Omega}^\mathrm{T} - I\mathbf{\Omega}\cdot\mathbf{\Omega})/4 - \mathscr{H}(p_\mathrm{c}) \\
&\quad = R_S(S_\mathrm{c}, \mathbf{\Omega}, \mathbf{u}_\mathrm{c}, p_\mathrm{c})
\end{aligned} \tag{17}$$

where $R_S(S_\mathrm{c}, \mathbf{\Omega}, \mathbf{u}_\mathrm{c}, p_\mathrm{c}) = -\dot{S}_\mathrm{c} - \mathbf{u}_\mathrm{c}\cdot\nabla S_\mathrm{c} - S_\mathrm{c}^\mathrm{T} S_\mathrm{c} + \nu\Delta S_\mathrm{c} - (\mathbf{\Omega}\mathbf{\Omega}^\mathrm{T} - I\mathbf{\Omega}\cdot\mathbf{\Omega})/4 - \mathscr{H}(p_\mathrm{c})$. Assuming that $p_\mathrm{f} = 0$, neglecting higher-order terms, and approximating the solution to this problem by the right-hand side, as above, we get

$$\boxed{S_\mathrm{f} = \tau_S R_S(S_\mathrm{c}, \mathbf{\Omega}, p_\mathrm{c})} \tag{18}$$

where we set $\tau_S = \tau_\omega = (c_1|\mathbf{u}_\mathrm{c}|/h + c_2\nu/h^2)$.

Here, we note the presence of the possibly destabilizing term $-S_\mathrm{c}^\mathrm{T} S_\mathrm{c}$, which gives rise to the additional driving term

$$-(S_\mathrm{c}\mathbf{u}_\mathrm{c}, S_\mathrm{c}\mathbf{v}_\mathrm{c}) \tag{19}$$

We also have

$$-((\mathbf{\Omega}\mathbf{\Omega}^\mathrm{T} - I\mathbf{\Omega}\cdot\mathbf{\Omega})\mathbf{u}_\mathrm{c}, \mathbf{v}_\mathrm{c}) = (\mathbf{u}_\mathrm{c}, \mathbf{v}_\mathrm{c})(\mathbf{\Omega}, \mathbf{\Omega}) - (\mathbf{u}_\mathrm{c}, \mathbf{\Omega})(\mathbf{v}_\mathrm{c}, \mathbf{\Omega}) = (\mathbf{u}_\mathrm{c}^\perp, \mathbf{v}_\mathrm{c}^\perp)(\mathbf{\Omega}, \mathbf{\Omega}) \tag{20}$$

which appears to give a certain control over $\mathbf{u}_\mathrm{c}^\perp$ the component of the velocity orthogonal to the vorticity, thus adding some stability. Moreover, numerical tests have shown that neglecting the term $S_\mathrm{f}^\mathrm{T} S_\mathrm{f}$ does not affect the stability significantly.

3.3. Evolution of fine-scale velocity

Following the same procedure as above, we get

$$\boxed{\mathbf{u}_\mathrm{f} = \tau_\mathbf{u} R_\mathbf{u}(\mathbf{u}_\mathrm{c}, p_\mathrm{c})} \tag{21}$$

where $1/\tau_\mathbf{u} = 1/\tau_\omega = 1/\tau_S = (c_1|\mathbf{u}_\mathrm{c}|/h + c_2\nu/h^2)$ and $R_\mathbf{u}(\mathbf{u}_\mathrm{c}, p_\mathrm{c}) = -\dot{\mathbf{u}}_\mathrm{c} - \mathbf{u}_\mathrm{c}\cdot\nabla\mathbf{u}_\mathrm{c} - \nabla p_\mathrm{c} + \nu\Delta\mathbf{u}_\mathrm{c}$ and where the fine-scale pressure gradient is neglected.

4. THE TAYLOR–GREEN VORTEX

The flow used here to study the properties of the presented scheme is the aforementioned Taylor–Green vortex which consists of incompressible flow in the cube $[0, 2\pi]^3$ with periodic boundary conditions in all three directions and initial conditions defined by (3). The *Re*-number in this study is $Re = 1600$. This flow is, despite its simplicity, challenging, since it acts as a fundamental prototype flow including phenomena such as vortex stretching and consequent production of small-scale eddies as well as the dynamics of transition to turbulence and the following decay. By tracking the development of kinetic energy and enstrophy we can test a scheme's capabilities to simulate the existence of an inertial sub-range in the kinetic energy spectra for sufficiently high *Re* numbers and the finite (viscosity independent) energy dissipation limit law. Here, the transitional phase for times $t \leqslant 10$ has been studied.

Figure 1 shows the temporal evolution of the flow in terms of iso-surfaces of the second largest eigenvalue of the velocity gradient, λ_2, at $t = 2.5$, 5, 7.5, and 10. A series of events leading to the transition and decay mentioned above have been identified for this problem, see e.g. [15, 18]. First, the initial quasi-inviscid evolution leads to the formation of vortex sheets and secondary

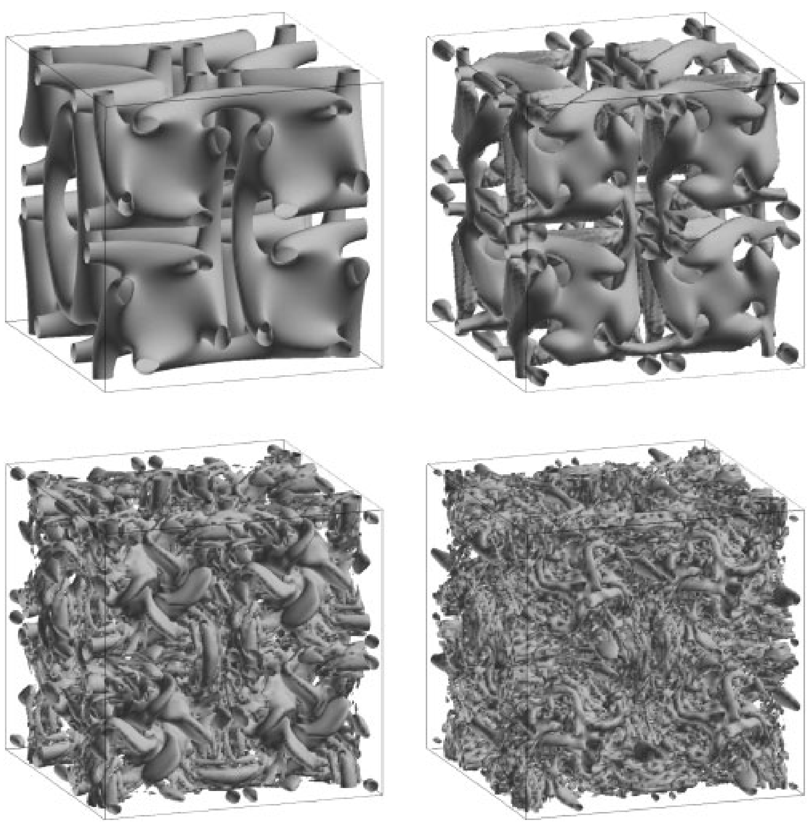

Figure 1. Isosurfaces of $\lambda_2 = \|\nabla \times u\| - \sqrt{\|\nabla u_x\|^2 + \|\nabla u_y\|^2 + \|\nabla u_z\|^2} = 0.25$, computed with $N = 96$, for $t = 2.5$, 5.0, 7.5, and 10.0.

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vortices around $t\approx 5$, which subsequently rolls-up and remain unchanged until $t\approx 6.5$. Thereafter, the rolled-up sheets break up and form distorted high-vorticity patches in a background of low vorticity. Furthermore, these high-vorticity (coherent structure) patches starts to break down at about $t\approx 8$ so that around $t\approx 9$ transition into turbulence occurs. This results in smaller-scale (but organized) vortices, which then slowly decays (after about $t\approx 10$) into a fully developed (disorganized) worm-vortex-dominated flow usually found in fully developed turbulence.

4.1. *Numerical methodology*

The presented vorticity–strain–VMS has been implemented in a finite-element setting using linear tetrahedral elements and time stepping is done *via* a conditionally stable operator splitting scheme, originally proposed by Ren and Utnes [19]. The algorithm consists of four steps: the calculation of an intermediate velocity field in two steps (omitting pressure and body forces), solution of a Poisson equation for the pressure (arising from the incompressibility constraint (2)), and a correction of the intermediate velocity field. The fine-scale terms (15), (18), and (21) are added as explicit forces to the resolved flow field. In order to study the behaviour of each of the three fine-scale terms, we multiply the different $\tau_i : s$ with a weighting parameter γ_i making it possible to amplify or dampen the influence of the fine-scale vorticity, strain and velocity, respectively. For stability reasons, this parameter was chosen to be $\gamma=\frac{1}{15}$; for larger γ the fine-scale terms showed a tendency to become too dominant, a problem we discuss further below.

4.2. *Computational results*

The Taylor–Green vortex flow described is computed up to $t=10$, with mesh resolutions of $\sim N^3+(N-1)^3$ for $N=64$ and 96 and at Reynolds number $Re=1/\nu=1600$. The DNS data provided by Brachet *et al.* [20] is used as a reference.

In Figure 2, the mean dissipation of kinetic energy, $-\mathrm{d}\langle K\rangle/\mathrm{d}t$, where $K=\frac{1}{2}u_i u_i$ is the kinetic energy, is plotted without any fine-scale modelling applied. We see that for $N=64$, the discrepancy

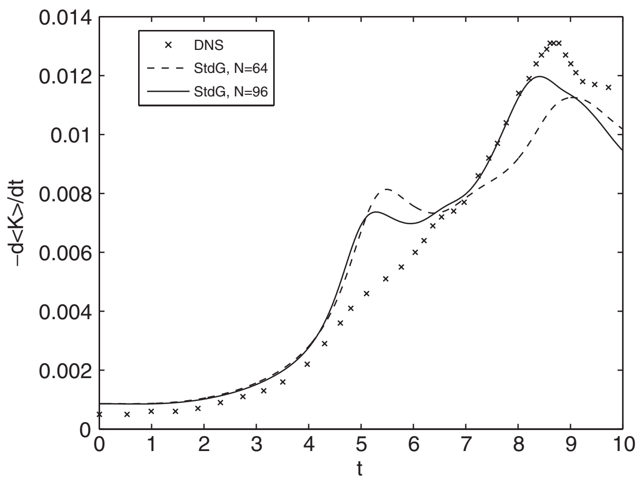

Figure 2. Mean dissipation of kinetic energy, $-\mathrm{d}\langle K\rangle/\mathrm{d}t$.

is large, mainly in three aspects. First, the initial dissipation is too high, secondly, a peak appears at $t \approx 5$ when the secondary vortices appear, and finally, the main peak at $t \approx 9$ appears too late and is too small. Switching to the higher resolution with $N = 96$, the initial dissipation is still too high, while the first peak is significantly reduced and the second one is considerably better both in time and in value. These problems occur also in other LES simulations of the Taylor–Green problem, see e.g. [18], but are not quite as severe. We note, however, that the first peak exists also in the DNS but is very small, more of a bump than a peak. Studying the behaviour of the flow for lower Re, where the transition to turbulence never occurs, it seems that this peak is the only one existing. This is also the case if even lower mesh resolution is attempted. Our conclusion is that the previously mentioned formation of vortex sheets and secondary vortices is at or beyond the actual resolution limit maybe even at $N = 96$ with a resulting increase in numerical diffusion. Once formed, the scheme recuperates until the breakdown of these structures causes the appearance of even smaller scales, which are not captured and thus the second peak is underestimated.

A previous study [21] has indicated that the derived fine-scale terms might have a tendency to drive the flow and even to destabilize it. The implications would be that when adding the subgrid vorticity, strain, and velocity, we might expect the discrepancy of the first peak to decrease but at the expense of worse prediction of the second peak. We see, however, that this is not the case. In Figure 3, the inclusion of the fine-scale terms yields a strong reduction of the first peak, while the second peak is clearly less affected. This implies that the slightly more detailed analysis implemented for this study improves the overall performance of the scheme.

For $N = 96$, the predicted dissipation is now in better agreement with DNS. For times between $t = 6$ and 8, where the intermediate-sized structures in the form of vortex sheets are beginning to roll up and break down, we have excellent agreement and the peak at $t \approx 9$ is underestimated by less than 10%. It is, however, disappointing that the initial dissipation is still slightly high. For the coarser mesh with $N = 64$, the first peak is reduced to the same level as initially could be seen on the finer resolution, but the prediction right after that peak is affected negatively. As a comparison, the standard one-equation eddy viscosity model by Shumann [22], which generally

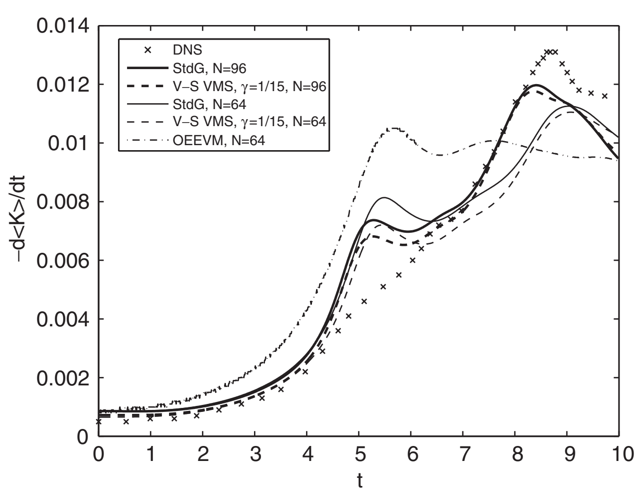

Figure 3. Mean dissipation, with and without the fine-scale terms. Included is also a computation with the one-equation eddy viscosity model on $N = 64$.

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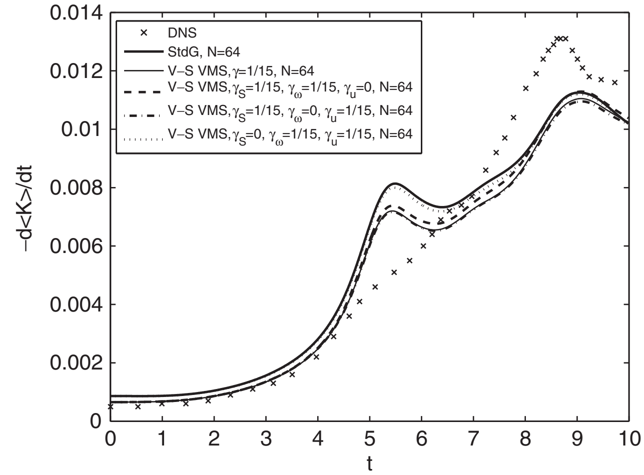

Figure 4. Mean dissipation, effect of the different fine-scale terms.

gives good results, see e.g. [23], is included in Figure 3. Unfortunately, computations with this model is at the moment only available on the coarse mesh with $N = 64$.

Studying the effect of each of the fine-scale terms, see Figure 4, we see that S_f has a strong driving effect before and around the first erroneous peak and is actually responsible for almost all of the improvement. This might be an indication that the operator splitting method implemented has problems enforcing the incompressibilty constraint, since, as mentioned above, in a divergence-free flow the strain can be expressed *via* the vorticity and the gradient of a scalar. This needs to be investigated further by implementing the fine-scale terms using another time-stepping scheme. If this is the reason, however, it shows also that this kind of residual-based turbulence modelling takes numerical defects of the low resolution into account. The fine-scale velocity is also a driving term, but more so around the second peak than the first. The effect is, however, much smaller than for the fine-scale strain. When it comes to $\mathbf{\Omega}_f$, this quantity has a slight dissipative behaviour during the entire simulation.

5. CONCLUSIONS

A VMS for LES, based on a vorticity–strain formulation of the Navier–Stokes equations, has been presented and applied to the initial, transitional part of the Taylor–Green vortex flow problem. The method leads to the addition of forces to the coarse-scale equations representing the interaction between fine-scale vorticity, strain and velocity, and coarse-scale velocity. The fine-scale quantities are driven by the residuals of the resolved flow field, thus taking both numerical and physical underresolution into account. In this study, the evolution on the fine scales is approximated with easily computed algebraic expressions.

The numerical tests show promising results with a significant correction of an erroneous dissipation peak at $t \approx 5$, where intermediate-sized flow structures are beginning to form, without damaging the prediction for later times. This indicates that the fine-scale addition counteracts the numerical diffusion while not disturbing the physical one. This is despite the much simplified expressions used. The problem with the first peak still remains though, but we doubt that there is room for much improvement, at least on coarser meshes. There it seems that there are energy containing structures that cannot be represented on the mesh, and this contradicts the basic assumption in LES that enough of the energetic structures need to be resolved. Moreover, the scheme does not capture the expected increased energy flow from resolved to subgrid scales for times $t \gtrsim 7$. According to Kerr *et al.* [14] a large amount of this energy exchange occurs in the terms involving u_{f}, and although we see the correct trend in our simulations it is much too weak.

The future development of the proposed VMS formulation includes time tracking of the fine-scale fields. This is necessary in order to include the history of the flow, which is likely to affect also the small scales. Lifting the assumption of quasi-stationary fine scales would remove some of the inconsistencies built into the τ_i parameters. Moreover, the high initial dissipation and the large first peak leads us to suspect that the predictor–corrector scheme used might not be the best choice. A better approach needs to be investigated. When it comes to correcting the lack of dissipation on the smallest scales, one could consider replacing the small–small-scale interaction, represented by the last term in (9), by a standard LES functional subgrid model. This can be compared with the so-called mixed subgrid models in the LES literature.

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